WAVELETS IN NUMERICAL ANALYSIS

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I INTRODUCTION

computing potential interactions has made explicit many of the ingredients of Calderón-Zygmund theory. In that paper a fast algorithm of order N to

[redacted]

$$V_i = \sum^{N} \frac{g_i g_j}{\ldots} \quad \text{where} \quad x_i \in \mathbf{R}^3, \; i, j = 1, \ldots, N$$

[redacted]

[redacted]

[redacted]

[redacted]

[redacted]

includes Calderón-Zygmund and pseudo-differential operators; these oper- [redacted]

lation and scale invariant size estimates). The numerical implementations described in this paper are the beginning of a program for the conversion of pseudo-differential calculus into a numerical tool. The main idea here is [redacted] where $\chi(x)$ is the characteristic function of the interval $(0,1)$ and $\chi_{j,k}(x) = 2^{-j/2}\chi(2^{-j}x-k)$. This basis leads to what we call the non-standard representation of an operator (the terminology will become clear later).

Third, we note that if we consider an integral operator

$$T(f)(x) = \int K(x,y)f(y)dy, \tag{2.5}$$

and expand its kernel in a two-dimensional Haar basis we find (for a wide [redacted]

from the diagonal is faster in these representations than that in the original [redacted]

[redacted]

In addition, the function ψ has M vanishing moments

$$\int_{-\infty}^{+\infty} \psi(x)x^m\,dx = 0, \qquad m = 0, \ldots, M-1. \tag{2.11}$$

[redacted]

of vanishing moments M. For the wavelets in [9] $L = 2M$. If additional [redacted] itly [8]. In Section VII we give an example of constructing the non-standard [redacted]

[redacted]

[redacted]

[redacted]

The $2M$-dimensional space $\mathbf{W}_{-1}^M$ is spanned by the $2M$ orthogonal functions $h_1(2x), \ldots, h_M(2x)$, $h_1(2x-1), \ldots, h_M(2x-1)$, of which M are supported on the interval $[0, \frac{1}{2}]$ and M on $[\frac{1}{2}, 1]$. In general, the space $\mathbf{W}_j^M$ is spanned by $2^{-j}M$ functions obtained from $h_1, \ldots, h_M$ by translation and dilation. There is some freedom in choosing the functions $h_1, \ldots, h_M$ within the constraint that they be orthogonal; by requiring normality and additional vanishing moments, we specify them uniquely (up to sign).

[illegible]

First let us construct M functions $f_1, \ldots, f_M : \mathbf{R} \to \mathbf{R}$ supported on the interval $[-1, 1]$ and such that

[illegible]

1. The restriction of f_i to the interval $(0, 1)$ is a polynomial of degree $M - 1$.

2. [illegible]

1. By the Gram-Schmidt process we orthogonalize f_m^1 with respect to $1, x, \ldots, x^{M-1}$, to obtain f_m^2, for $m = 1, \ldots, M$. This orthogonality is preserved by the remaining orthogonalizations, which only produce linear combinations of the f_m^2.

2. The following sequence of steps yields $M-1$ functions orthogonal to x^M, of which $M-2$ functions are orthogonal to x^{M+1}, and so forth, down to one function which is orthogonal to x^{2M-2}. First, if at least one of f_m^2 is not orthogonal to x^M, we reorder the functions so that it appears first, $\langle f_1^2, x^M \rangle \neq 0$. We then define $f_m^3 = f_m^2 - a_m \cdot f_0^2$ [redacted] ing the desired orthogonality to x^M. Similarly, we orthogonalize to [redacted] $x^{M+1}, \ldots, x^{2M-2}$, each in turn, to obtain $f_1^2, f_2^3, f_3^4, \ldots, f_M^{M+1}$ such that $\langle f_m^{m+1}, x^i \rangle = 0$ for $i \leq m + M - 2$.

3. Finally, we do Gram-Schmidt orthogonalization on $f_M^{M+1}, f_{M-1}^M, \ldots, f_1^2$, in that order, and normalize to obtain $f_M, f_{M-1}, \ldots, f_1$.

[redacted]

where $\mathbf{V}_j^M$ is given in (3.1), and the space $\mathbf{W}_j^{M,2}$ as the orthogonal complement of $\mathbf{V}_j^{M,2}$ in $\mathbf{V}_{j-1}^{M,2}$,

$$\mathbf{V}_{j-1}^{M,2} = \mathbf{V}_j^{M,2} \bigoplus \mathbf{W}_j^{M,2}.$$

[redacted]

$$\{u_i(x)h_l(y),\ h_i(x)u_l(y),\ h_i(x)h_l(y):\ \ i,l = 1,\ldots,M\}.$$

[redacted]

On each fixed scale j, the wavelets $\{\psi_{j,k}(x)\}$ ... form an ...

where $T \sim T_0 = P_0 T P_0$ is a discretization of the operator T on the finest scale.

The non-standard form is a representation (see [5]) of the operator T as a chain of triplets

$$T = \{A_j, B_j, \Gamma_j\}_{j \in \mathbf{Z}} \tag{4.14}$$

acting on the subspaces $\mathbf{V}_j$ and $\mathbf{W}_j$

$$A_j : \mathbf{W}_j \to \mathbf{W}_j, \tag{4.15}$$

$$B_j : \mathbf{V}_j \to \mathbf{W}_j, \tag{4.16}$$

$$\Gamma_j : \mathbf{W}_j \to \mathbf{V}_j. \tag{4.17}$$

The operators $\{A_j, B_j, \Gamma_j\}_{j \in \mathbf{Z}}$ are defined as $A_j = Q_j T Q_j$, $B_j = Q_j T P_j$ and $\Gamma_j = P_j T Q_j$.

The operators $\{A_j, B_j, \Gamma_j\}_{j \in \mathbf{Z}}$ admit a recursive definition (see [5]) via

the relation

$$T_j = \begin{pmatrix} A_{j+1} & B_{j+1} \\ \Gamma_{j+1} & T_{j+1} \end{pmatrix}, \tag{4.18}$$

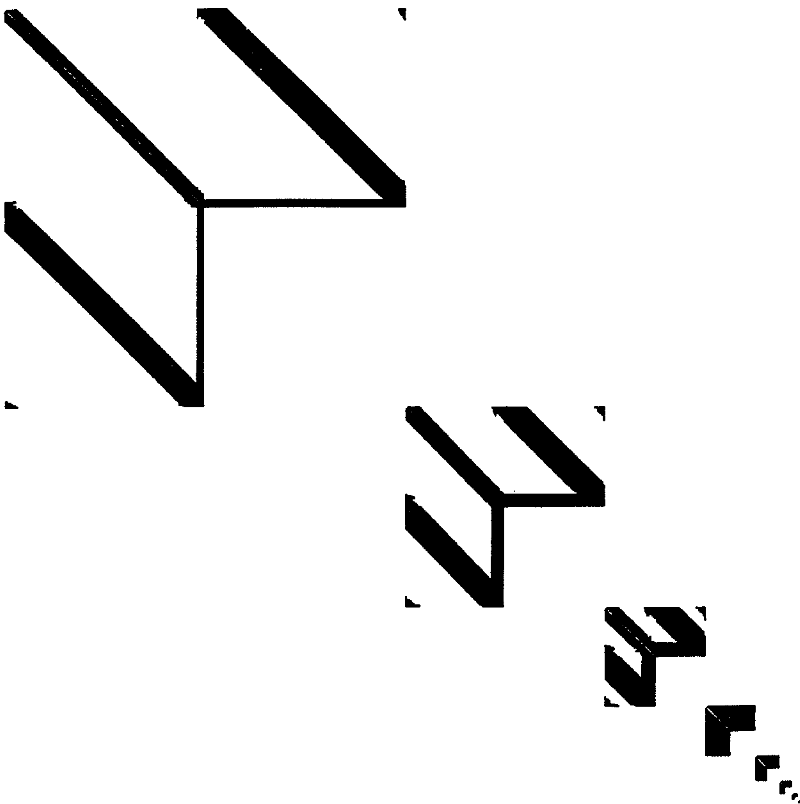
where operators $T_j = P_j T P_j$,

$$T_j : \mathbf{V}_j \to \mathbf{V}_j. \tag{4.19}$$

If there is a coarsest scale n, then

$$T = \{\{A_j, B_j, \Gamma_j\}_{j \in \mathbf{Z}: j \leq n}, T_n\}, \tag{4.20}$$

where $T_n = P_n T P_n$. If the number of scales is finite, then $j = 1, 2, \ldots, n$



[redacted]

$$\beta^j_{i,l} = \sum_{k,m=0}^{L-1} g_k h_m s^{j-1}_{k+2i,m+2l}, \tag{4.26}$$

$$\gamma^j_{i,l} = \sum_{k,m=0}^{L-1} h_k g_m s^{j-1}_{k+2i,m+2l}, \tag{4.27}$$

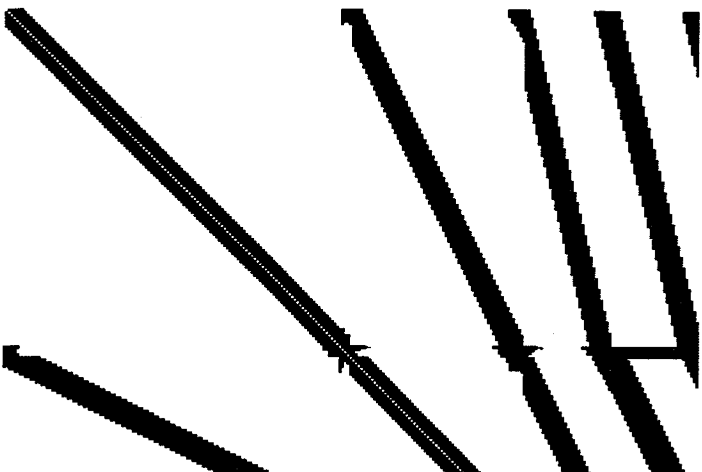
$$s^j_{i,l} = \sum_{k,m=0}^{L-1} h_k h_m s^{j-1}_{k+2i,m+2l}, \tag{4.28}$$

with $i, l = 0, 1, \ldots, 2^{n-j} - 1$, $j = 1, 2, \ldots, n$. Clearly, formulae (4.25) – [illegible]

[illegible]

matrices $\alpha^j, \beta^j, \gamma^j$ with $j = 1, 2 \ldots, n$.

To compute the coefficients $s^0_{k,k'}$, we refer to [5], where wavelet-based quadratures for the evaluation of these coefficients are developed. Also, we refer to [5] for a fast algorithm (order N) for constructing the non-standard form for operators with known singularities, and to [8] for the direct eval-



There are two ways of computing the standard form of a matrix. First consists in applying the one-dimensional transform (see (2.12) and (2.13))

$$| \partial_\xi^\alpha \, \partial_x^\beta \, \sigma^*(x,\xi) | \leq C_{\alpha,\beta}(1+ | \xi |)^{\lambda-\alpha+\beta}, \tag{6.8}$$

[redacted]

$$|\alpha^j| + |\beta^j| + |\gamma^j| \leq \frac{2^{\lambda j} \, C_M}{\ldots} \tag{6.9}$$

[redacted]

for all integer i, l.

Suppose now that we approximate the operator T_0 by the operator T_0^B [redacted]

$$\gamma(y) = T^*(1)(y) \tag{6.15}$$

belong to dyadic $B.M.O.$, i.e. satisfy condition

$$\sup_J \frac{1}{|J|} \int_J |\beta(x) - m_J(\beta)|^2 dx \le C, \tag{6.16}$$

where J is a dyadic interval and

$$m_J(\beta) = \frac{1}{|J|} \int_J \beta(x) dx. \tag{6.17}$$

Again we refer to [5] for details.

VII THE OPERATOR d/dx IN WAVELET BASES

As an example, we construct the non-standard form of the operator d/dx [8]. The matrix elements α^j_{il}, β^j_{il}, and γ^j_{il} of A_j, B_j, and Γ_j, where $i, l, j \in \mathbf{Z}$ for the operator d/dx are easily computed as

$$\alpha^j_{il} = 2^{-j} \int_{-\infty}^{\infty} \psi(2^{-j}x - i)\, \psi'(2^{-j}x - l)\, 2^{-j} dx = 2^{-j} \alpha_{i-l}, \tag{7.1}$$

[illegible]

and

$$\gamma^j_{il} = 2^{-j} \int_{-\infty}^{\infty} \varphi(2^{-j}x - i)\, \psi'(2^{-j}x - l)\, 2^{-j} dx = 2^{-j} \gamma_{i-l}, \tag{7.3}$$

$$\beta_i = 2 \sum_{k=0}^{L-1} \sum_{k'=0}^{L-1} g_k \, h_{k'} \, r_{2i+k-k'}, \tag{7.8}$$

and

$$\gamma_i = 2 \sum_{k=0}^{L-1} \sum_{k'=0}^{L-1} h_k \, g_{k'} \, r_{2i+k-k'}, \tag{7.9}$$

where

$$r_l = \int_{-\infty}^{+\infty} \varphi(x-l) \, \frac{d}{dx} \varphi(x) \, dx, \quad l \in \mathbf{Z}. \tag{7.10}$$

Therefore, the representation of d/dx is completely determined by r_l in (7.10) or in other words, by the representation of d/dx on the subspace $\mathbf{V}_0$.

[illegible]

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(x) \, \mathrm{e}^{\mathrm{i}x\xi} \, dx, \tag{7.11}$$

we obtain

$$r_l = \int_{-\infty}^{+\infty} |\hat{\varphi}(\xi)|^2 (\mathrm{i}\xi) \mathrm{e}^{-\mathrm{i}l\xi} \, d\xi. \tag{7.12}$$

[illegible]

and

$$r_l = -r_{-l}, \tag{7.16}$$

Solving equations (7.13), (7.14), we present the results for Daubechies' wavelets with $M = 2, 3, 4, 5$. For further examples we refer to [8].

1. $M = 2$

$$a_1 = \frac{9}{8}, \quad a_3 = -\frac{1}{8},$$

and

$$r_1 = -\frac{2}{3}, \quad r_2 = \frac{1}{12},$$

We note, that the coefficients $(-1/12, 2/3, 0, -2/3, 1/12)$ of this example can be found in many books on numerical analysis as a choice of coefficients for numerical differentiation.

2. $M = 3$

$$a_1 = \frac{75}{64}, \quad a_3 = -\frac{25}{128}, \quad a_5 = \frac{3}{128},$$

and

$$r_1 = -\frac{272}{[illegible]}, \quad r_2 = \frac{53}{[illegible]}, \quad r_3 = -\frac{16}{[illegible]}, \quad r_4 = -\frac{1}{[illegible]}.$$

3. $M = 4$

$$a_1 = \frac{1225}{1024}, \quad a_3 = -\frac{245}{1024}, \quad a_5 = \frac{49}{1024}, \quad a_7 = -\frac{5}{1024},$$

and

$$r_1 = -\frac{39296}{49553}, \quad r_2 = \frac{76113}{396424}, \quad r_3 = -\frac{1664}{49553},$$

$$r_4 = \frac{2645}{[illegible]}, \quad r_5 = \frac{128}{[illegible]}, \quad r_6 = -\frac{1}{[illegible]}.$$

$$r_4 = \frac{17297069}{2318208034}, \quad r_5 = -\frac{1386496}{5795520085}, \quad r_6 = -\frac{563818}{10431936153},$$

$$r_7 = -\frac{2048}{8113728119}, \quad r_8 = -\frac{5}{18545664272}.$$

Remark 1. If $M = 1$, then equations (7.13) and (7.14) have a unique solution but the integrals in (7.10) or (7.12) may not be absolutely convergent. For the Haar basis ($h_1 = h_2 = 2^{-1/2}$) $a_1 = 1$ and $r_1 = -1/2$ and we obtain the simplest finite difference operator $(1/2, 0, -1/2)$. In this case the function φ is not continuous and

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{\sin \frac{1}{2}\xi}{\frac{1}{2}\xi} \, \mathrm{e}^{\mathrm{i}\frac{1}{2}\xi}.$$

Remark 2. For the coefficients $r_l^{(n)}$ of d^n/dx^n, $n > 1$, the system of linear algebraic equations is similar to that for the coefficients of d/dx. This system (and (7.13)) may be written in terms of

$$\hat{r}(\xi) = \sum_l r_l^{(n)} \mathrm{e}^{\mathrm{i}l\xi}, \tag{7.17}$$

as

$$\hat{r}(\xi) = 2^n \left(|m_0(\xi/2)|^2 \, \hat{r}(\xi/2) + |m_0(\xi/2 + \pi)|^2 \, \hat{r}(\xi/2 + \pi) \right), \tag{7.18}$$

where m_0 is the 2π-periodic function

$$m_0(\xi) = 2^{-1/2} \sum_{k=0}^{k=L-1} h_k \mathrm{e}^{\mathrm{i}k\xi}, \tag{7.19}$$

and h_k are the wavelet coefficients. Considering the operator M_0 on 2π-periodic functions

$$(M_0 f)(\xi) = |m_0(\xi/2)|^2 \, f(\xi/2) + |m_0(\xi/2 + \pi)|^2 \, f(\xi/2 + \pi), \tag{7.20}$$

we rewrite (7.18) as

$$M_0 \hat{r} = 2^{-n} \hat{r}, \tag{7.21}$$

so that $\hat{r}$ is an eigenvector of the operator M_0 corresponding to the eigenvalue 2^{-n}. Thus, finding the representation of the derivatives in the wavelet basis is equivalent to finding trigonometric polynomial solutions of (7.21) and vice versa [8].

Remark 3. While theoretically it is well understood that the operaand $\hat{T}$. For example,

$$\Gamma_1^2 = \tilde{\Gamma}_1^2 \hat{A}_1 + \tilde{A}_2 \hat{\Gamma}_1^2 + \sum_{j'=3}^{j'=n+1} \tilde{B}_2^{j'} \hat{\Gamma}_1^{j'}. \tag{8.4}$$

[redacted]

erators, then all the blocks of (8.1) and (8.2) (except for $\tilde{T}_n$ and $\hat{T}_n$) are

[redacted]

[redacted]

$$X_0 = \alpha A^*, \tag{9.2}$$

where A^* is the adjoint matrix and α is chosen so that the largest eigen- [redacted]

generalized inverse $A^\dagger$.

When this result is combined with the fast multiplication algorithm of where $i, j = 1, \ldots, N$. The accuracy theshold was set to 10^{-4}, i.e., entries of X_k below 10^{-4} were systematically removed after each iteration.

X SOME PRELIMINARY RESULTS AND DIRECTIONS OF RESEARCH

In this section we describe several iterative algorithms indicating that numerical functional calculus with operators can be implemented efficiently [illegible]

performance of these algorithms will be reported separately.

Remark on iterative computation of the projection operator on the null space.

We present here a fast iterative algorithm for computing P_{null} for a wide class of operators compressible in the wavelet bases.

Let us consider the following iteration

$$X_{k+1} = 2X_k - X_k^2 \tag{10.1}$$

with

with

$$Y_0 = \frac{\alpha}{2}(A+I),$$
$$X_0 = \frac{\alpha}{2}(A+I), \qquad (10.5)$$

where α is chosen so that the largest eigenvalue of $\frac{\alpha}{2}(A+I)$ is less than $\sqrt{2}$.

The sequence X_l converges to $A^{1/2}$ and Y_l to $A^{-1/2}$. By writing $A =$ [redacted]

both X_l and Y_l can be written as $X_l = V^* P_l V$ and $Y_l = V^* Q_l V$, where P_l and Q_l are diagonal and

$$Q_{l+1} = 2Q_l - Q_l P_l Q_l,$$

[redacted]

with

$$Q_0 = \frac{\alpha}{2}(D+I),$$

one. At the second stage of the algorithm the matrix $2^{-L}A$ is squared L times to obtain the result.

Similarly, sine and cosine of a matrix can be computed using the elementary double-angle formulas. On denoting

$$Y_l = \cos(2^{l-L}A) \tag{10.9}$$

$$X_l = \sin(2^{l-L}A), \tag{10.10}$$

we have for $l = 0, \ldots, L-1$

$$Y_{l+1} = 2Y_l^2 - I \tag{10.11}$$

$$X_{l+1} = 2Y_l X_l,, \tag{10.12}$$

where I is the identity. Again, we choose L so that the largest singular value of $2^{-L}A$ is less than one, compute the sine and cosine of $2^{-L}A$ using the Taylor series, and then use (10.11) and (10.12).

[redacted]

number of multiplications of dense matrices has to be performed [19]. Fast multiplication algorithm of Section VIII reduces complexity to not more than $O(N \log^2 N)$ operations.

To acheive such perfomance it is necessary to maintain the "finger" band structure of the standard form throughout the iteration. Whether it is possible to do depends on the particular operator and, usually, can be

[redacted]

[5] G. Beylkin, R. R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms I.* Yale University Technical Report [redacted]

[18] G. Beylkin, R. R. Coifman and V. Rokhlin, *Fast wavelet transforms and numerical algorithms II.*, in progress.

[19] R. C. Ward, *Numerical computation of the matrix exponential with accuracy estimates*, SIAM J. Numer. Anal. vol.14, 4, 600-610, 1977.