Applied Analysis Preliminary Exam

10.00am–1.00pm, August 21, 2017 (Draft v7, Aug 20)

Instructions. You have three hours to complete this exam. Work all five problems. Please start each problem on a new page. Please clearly indicate any work that you do not wish to be graded (e.g., write SCRATCH at the top of such a page). You MUST prove your conclusions or show a counter-example for all problems unless otherwise noted. In your proofs, you may use any major theorem on the syllabus or discussed in class, unless you are directly proving such a theorem (when in doubt, ask the proctor). Write your student number on your exam, not your name. Each problem is worth 20 points. (There are no optional problems.)

Problem 1:

(a) Let $\mathcal{F}$ be a family of equicontinuous functions from a metric space $(X; d_X)$ to a metric space $(Y; d_Y)$. Show that the completion of $\mathcal{F}$ is also equicontinuous.

(b) Let $(f_n)_{n\geq 1}$ be a sequence of functions in $C([0;1])$. Let $\|\cdot\|$ be the sup norm. Suppose that, for all n, we have

- $\|f_n\| \leq 1$,
- f_n is differentiable, and
- $\|f_n'\| \leq M$ for some $M \geq 0$.

Show that the completion of $\{f_n\}_{n\geq 1}$ is compact, and therefore that it has a convergent subsequence.

Problem 2:

Show that there is a continuous function u on $[0;1]$ such that

$$u(x) = x^2 + \frac{1}{8}\int_0^x \sin(u^2(y))\,dy.$$

Problem 3:

Let $f \in L^\infty(\mathbb{R})$. Show that

$$\lim_{n\to\infty}\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}\,dx\right)^{1/n}$$

exists and equals $\|f\|_\infty$.

Problem 4:

Let $K : L^2([0;1]) \; ! \; L^2([0;1])$ be the integral operator de ned by

$$Kf(x) = \int_0^x f(y)\,dy:$$

This operator can be shown to be compact by using the Arzela-Ascoli Theorem. For this problem, you may take compactness as fact.

(a) Find the adjoint operator K of K.

(b) Show that $jjKjj^2 = jjK \; Kjj$.

(c) Show that $jjKjj = 2=$. (Hint: Use part (b).)

(d) Prove that

$$K^n f(x) = \frac{1}{(n \quad 1)!} \int_0^x f(y)(x \quad y)^{n \;\; 1}\,dy:$$

(e) Show that the spectral radius of K

Problem 1 Solution:

(a) This part is almost trivial. It is just here to help with part (b).
Recall that F being equicontinuous means that, for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$ holds $\forall f \in F$.
To show equicontinuity of the completion, we need only worry about the additional included functions. Let g be a function in the completion of F that was not in F to begin with. Since F is dense in the completion, we can find an $f \in F$ that is arbitrarily close to g. In particular, chose $f \in F$ such that $d_Y(f(x), g(x)) < \varepsilon/3$ $\forall x \in X$.
Let $\varepsilon > 0$. Note that

$$d_Y(g(x), g(y)) \le d_Y(g(x), f(x)) + d_Y(f(x), f(y)) + d_Y(f(y), g(y)).$$

Since $f \in F$, we can find a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon/3$ and we are done. This δ gives us $d_X(x, y) < \delta \Rightarrow d_Y(g(x), g(y)) < \varepsilon$.

(b) We will use the Arzela-Ascoli Throrem: Let K be a compact metric space. A subset of $C(K)$ is compact if and only if it is closed, bounded, and equicontinuous.
The completion of $\{f_n\}$, is, by definition, closed
By the assumptions of this problem, we also have that the completion of $\{f_n\}$ is bounded.
It remains to show that the completion of $\{f_n\}$ is equicontinuous.
Take $\varepsilon > 0$. Fix n. By the Intermediate Value Theorem, we know that, $\forall\, x, y \in [0, 1]$, there exists a c between x and y such that $f_n(x) - f_n(y) = f_n'(c)(x - y)$.
Thus, we have that $f_n(x) - f_n(y) \le M|x - y|$.
Define $\delta = \varepsilon/M$. We then have

$$|x - y| < \delta \Rightarrow |f_n(x) - f_n(y)| < \varepsilon.$$

Note that this is independent of the choice of n.
Thus, the family of functions $\{f_n\}$ is equicontinuous.
By part (a) we know then that the completion of this family is equicontinuous.
By the Arzela-Ascoli Throrem, we then have that the completion of $\{f_n\}$ is compact, as desired.

Problem 2 Solution:

We will use the Contraction Mapping Theorem: If $T : X \to X$ is a contraction mapping on a complete metric space (X, d), then T has exaclty one fixed point. (i.e. There is exactly one $x \in X$ such that $T(x) = x$.)
Define

$$Tu(x) = x^2 + \frac{1}{8}\int_0^x \sin(u^2(y))\, dy.$$

Note that T maps $C([0, 1])$ functions to $C([0, 1])$ functions. Since $C([0, 1])$ is complete with respect to the sup norm $\|\ \|_\infty$, the contraction mapping theorem aorem aorem aorem u.504how tha

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By the mean value theorem, we know that there is some $s \in [0,1]$ such that

$$\frac{\sin u - \sin v}{u - v} \le \cos s \le 1$$

so

$$|\sin u(y) - \sin v(y)| \le |u(y) - v(y)|.$$

So, we have that

$$\begin{aligned} \|Tu - Tv\|_\infty &\le \tfrac{1}{8} \sup_{0 \le x \le 1} \int_0^x |u^2(y) - v^2(y)|\,dy \\ &= \tfrac{1}{8} \sup_{0 \le x \le 1} \int_0^x |u(y) + v(y)|\,|u(y) - v(y)|\,dy \\ &\le \tfrac{1}{8} \sup_{0 \le x \le 1} \int_0^x (|u(y)| + |v(y)|)\,|u(y) - v(y)|\,dy \end{aligned}$$

Since u and v are assumed to be continuous functions on the closed bounded interval $[0,1]$, they are bounded on $[0,1]$. Suppose that they are bounded by $M > 0$. Then

$$\begin{aligned} \|Tu - Tv\|_\infty &\le \tfrac{2M}{8} \sup_{0 \le x \le 1} \int_0^x |u(y) - v(y)|\,dy \\ &\le \tfrac{M}{4} \int_0^1 |u(y) - v(y)|\,dy \le \tfrac{M}{4} \|u - v\|_\infty \int_0^1 dy \\ &= \tfrac{M}{4} \|u - v\|_\infty \end{aligned}$$

This may or may not be a contraction, depending on the value of M, but, we are trying to show **existence** of a solution in $C([0,1])$. If we can show existence of a solution on some subset of $C([0,1])$, we are done. So, let's limit our search to the set of continuous functions on $[0,1]$ that are bounded, in the uniform norm, by some fixed constant M such that $M < 4$. Fix such an M and define the space

$$\mathcal{C} := \{u \in C([0,1]) : \|u\|_\infty \le M\} \subseteq C([0,1]).$$

Note that this is a closed (and non-empty!) subset of the complete $C([0,1])$ and is therefore complete. Furthermore, M can be chosen so that $T : \mathcal{C} \to \mathcal{C}$.
Thus, we have a contraction maping on a complete space (that is a subspace of the space of interest). By the Contraction Mapping Theorem, there exists a unique fixed point $u \in \mathcal{C} \subseteq C([0,1])$, which is a solution to the problem.

Problem 3 Solution:
This is trivial if $\|f\|_\infty = 0$. So, let us consider the case where $\|f\|_\infty > 0$.
Note that

$$\left(\int_{\mathbb{R}} \frac{|f(x)|^n}{1 + x^2}\,dx\right)^{1/n} \le \|f\|_\infty \left(\int_{\mathbb{R}} \frac{1}{1 + x^2}\,dx\right)^{1/n} = \|f\|_\infty\, \pi^{1/n} \to \|f\|_\infty \qquad \text{(S1)}$$

as $n \to \infty$.
On the other hand, by definition of $\|f\|_\infty$, for any $0 < \varepsilon < \|f\|_\infty$, there exists an $A \subseteq \mathbb{R}$ (with positive Lebesgue measure) such that $|f(x)| > \|f\|_\infty - \varepsilon \;\forall x \in A$.
Thus, we have

$$\int_{\mathbb{R}} \frac{|f(x)|^n}{1 + x^2}\,dx \ge \int_A \frac{|f(x)|^n}{1 + x^2}\,dx \ge (\|f\|_\infty - \varepsilon)^n \int_A \frac{1}{1 + x^2}\,dx.$$

Note that $\int_A \frac{1}{1+x^2}\,dx$ is strictly positive. Call it $c > 0$.

For all n, we now have

$$\int_{\mathbb{R}} \frac{|f(x)|^n}{1+x^2}$$

(c)

so we have that

$$[(n-1)!]^{1/n} \geq (\sqrt{2\pi})^{1/n}(n-1)^{1-1/(2n)}e^{1/n-1}$$

which goes to ∞ as $n \to \infty$.

In conclusion, the spectral radius is

$$r(K) = \lim_{n \to \infty} ||K^n||^{1/n} = 0;$$

as desired.

Problem 5 Solution: