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Higher-order networks[1–4] are attracting increasing attention as they are able to capture the many-body interactions of complex systems ranging from brain to social networks. Simplicial complexes are higher-order networks that encode the network geometry and topology of real datasets. Using simplicial complexes allows the network scientist to formulate new mathematical frameworks for mining data[5–10] and for understanding these generalized network structures revealing the underlying deep physical mechanisms for emergent geometry[11–15] and for higher-order dynamics[16–33]. In particular, this very vibrant research activity is relevant in neuroscience to analyze real brain data and its profound relation to dynamics[1,6,15,34–37] and in the study of biological transport networks[10,38].

In networks, dynamical processes are typically defined over signals associated to the nodes of the network. In particular, the Kuramoto model[39–43] investigates the synchronization of phases associated to normal distribution $\omega \sim \mathcal{N}(\Omega_0, 1/\tau_0)$. In absence of any by the model is different. In this case the dynamical equations are taken to be

$$\dot{\theta} = \omega - \sigma_1^{\mathrm{down}} B_{[1]} \sin(B_{[1]}^{\top}\theta), \tag{16}$$

$$\begin{aligned}\dot{\phi} = \tilde{\omega} &- \sigma_0{}_1^{\mathrm{up}} B_{[1]}^{\top} \sin(B_{[1]}\phi) \\ &- \sigma_1^{\mathrm{down}} B_{[2]} \sin(B_{[2]}^{\top}\phi).\end{aligned} \tag{17}$$

For Model NLT the projected dynamics for $\phi^{[-]}$ and for $\phi^{[+]}$ obeys

$$\dot{\phi}^{[-]} = B_{[1]}\tilde{\omega} - \sigma_0{}_1^{\mathrm{up}} L_{[0]} \sin\phi^{[-]}, \tag{18}$$

$$\dot{\phi}^{[+]} = B_{[2]}^{\top}\tilde{\omega} - \sigma_1^{\mathrm{down}} L_{[2]}^{\mathrm{down}} \sin\phi^{[+]}. \tag{19}$$

Therefore, as in Model NL, the dynamics of the projection $\phi^{[-]}$ of the phases ϕ associated to the links [Eq. (18)] is coupled to the dynamics of the phases θ associated directly to nodes [Eq. (16)] and vice versa. Moreover, the dynamics of the projection of the phases ϕ

zero. In fact

$$\sum_{¼1} \hat{\omega} \; ¼ \; {}^{T}\omega$$

numerical solution of Eq. (59) reveals the following picture: for low values of σ, only the incoherent solution ${}_0 = {}_1^{\mathrm{down}} = 0$ exists. At a positive value of σ, two solutions of Eq. (59) appear at a bifurcation point, with the upper solution corresponding to a stable synchronized state and the lower solution to an unstable synchronized solution. For larger values of σ, the values of ${}_0$ and ${}_1^{\mathrm{down}}$ corresponding to the upper solution approach one (full phase synchronization), while those for the lower solution approach zero asymptotically, thus indicating that the incoherent state never loses stability. Indeed, it can be easily checked (see "Methods" for details) that for large σ the unstable solution of Eq. (59) has asymptotic behavior

$$\begin{aligned} {}_0 &= \sigma^{-2}\,{}_0, \\ {}_1^{\mathrm{down}} &= \sigma^{-1}\,{}_1, \end{aligned} \qquad (61)$$

with ${}_0$ and ${}_1$ constants given by

$${}_0 = \frac{\pi}{2}$$

topologies can sustain a non-trivial hysteresis loop we expand Eq. (57) for $0 < {}_0 \ll 1$, $0 < \hat{}_0 \ll 1$, and $0 < {}_1^{down} \ll 1$ under the hypothesis that the distributions (ω) and $(\hat{\omega})$ are symmetric and unimodal. Under these hypothesis it is easy to show that Eq. (57) predict an unstable solution in which ${}_0$ and ${}_1^{down}$ scale with σ according to

$$
\begin{aligned}
{}_0 &= \sigma^{-2}\, {}_0, \\
{}_1^{down} &= \sigma^{-1}\, {}_1,
\end{aligned}
\qquad (64)
$$

with ${}_0$ and ${}_1$ constants given by

$$
\begin{aligned}
{}_0 &= \langle \ \rangle \left[\frac{\pi}{2}\frac{\langle {}^2 \rangle}{\langle \ \rangle}\right]^{-2} \underbrace{(\Omega_0)\frac{1}{\ }\sum \ , \ (\langle \hat{\omega} \rangle)}^{-1}, \\
{}_1 &= \left[(\Omega_0)\frac{\pi}{2}\frac{\langle {}^2 \rangle}{\langle \ \rangle}\right]^{-1}.
\end{aligned}
\qquad (65)
$$

As long as the network does not have vanishing ${}_0$ and ${}_1$ the unstable branch converges to the trivial solution ${}_0 = {}_1^{down}$ only in the limit $\sigma \to \infty$. This happens for instance for Gaussian distribution of the internal frequency of the links and converging second moment follows that the incidence matrices obey

$$\mathbf{B}_{[n]}\mathbf{B}_{[n+1]} = \mathbf{0}, \quad \mathbf{B}^{\top}_{[n+1]}\mathbf{B}^{\top}_{[n]} = \mathbf{0}, \tag{73}$$

for any $n > 0$.

Higher-order Laplacians. Using the incidence matrices it is natural to generalize the definition of the graph Laplacian

$$\mathbf{L}_{[0]} = \mathbf{B}_{[1]}\mathbf{B}^{\top}_{[1]} \tag{74}$$

to the higher-order Laplacian $\mathbf{L}_{[n]}$ (also called combinatorial Laplacians)[17,19,60] that can be represented as a $N_{[n]} \times N_{[n]}$ matrix given by

$$\mathbf{L}_{[n]} = \mathbf{L}^{\mathrm{down}}_{[n]} + \mathbf{L}^{\mathrm{up}}_{[n]} \tag{75}$$

with

$$\begin{aligned} \mathbf{L}^{\mathrm{down}}_{[n]} &= \mathbf{B}^{\top}_{[n]}\mathbf{B}_{[n]}, \\ \mathbf{L}^{\mathrm{up}}_{[n]} &= \mathbf{B}_{[n+1]}\mathbf{B}^{\top}_{[n+1]}, \end{aligned} \tag{76}$$

for $n > 0$. The higher-order Laplacian can be proven to be independent of the orientation of the simplices as long as the simplicial complex has an orientation induced by a labeling of the nodes.

References

1. Giusti, C., Ghrist, R. & Bassett, D. S. Two

Competing interests

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Additional information

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