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LTHOUGH a sphere is an ubiquitous object, constructing a local basis on it has proven difficult. The basis functions most commonly used for a sphere are the spherical harmonics. One solution of Laplace's equation uses spherical harmonics to solve a boundary-value problem on the surface of a sphere. A solution in the spherical system of coordinates is used to construct geopotential models, such as

$$U\ r;\ ;\qquad \frac{}{r}\left(1\quad \sum_{n\ 2}^{1}\sum_{m\ 0}^{n}\left(\frac{R}{r}\right)^{n} P_{n;m}\qquad C_{n;m}\qquad m \qquad S_{n;m}\qquad m\right) \tag{1}$$

where r, , and are the radius, geocentric latitude, and longitude, respectively; is the gravitation parameter; R is the radius of the primary body; $P_{n;m}$ is the associated Legendre polynomial of degree and order n and m; and the coefficients $C_{n;m}$ and $S_{n;m}$ describe the geopotential model. Gravitational acceleration, which is required for most applications, is found by evaluating r U.

When using the spherical harmonics, model accuracy improves by increasing the degree and order. As demand for improved gravity-model accuracy increases, so do the computational resources required for model evaluation. Additionally, orbits about bodies with irregular mass distributions, such as the moon, require a high-degree model to properly propagate an orbit [1]. Unfortunately, an increase in the degree and order of the model by a factor of 10 results in computation time increasing by a factor of 100 [2]. Interpolation models have been developed to make evaluation faster. Some of these models preserve the spherical coordinate system [3,4], while others drastically reformulate the evaluation of the gravity field [5,6].

Each term of the spherical harmonic model describes a variation in the geopotential mapped over the complete sphere. For example, the J_2 term describes the equatorial deviation from a sphere for all longitudes. Hence, each term is part of a global model. Unfortunately, the spherical harmonic model is unable to meet the demands for regional representations [7]. Several alternative methods have been explored to localize the gravity field for these scientific applications [8,9].

A new model, the cubed sphere, was developed to localize the representation of the gravity field and decrease the model evaluation time [2]. At its core, the cubed sphere is an interpolation model that relies on a localized representation defined on the surface of a segmented cube. We explore applications of the cubed-sphere model to orbit propagation: particularly, how it compares with the spherical harmonic model solutions.

Originally proposed by Beylkin and Cramer [2], the cubed-sphere model defines a new method to compute geopotential and acceleration. Essentially, the sphere is mapped to a cube with a new coordinate system defined on each face. Each face is segmented by a uniform grid, and interpolation is performed to find the acceleration. Multiple spheres, each mapped to a cube, are nested within each other, and interpolation is performed between adjacent shells to account for the acceleration variation in the radial direction. The mapping of a sphere to a cube is illustrated in Fig. 1. A grid-spacing scheme is established, with values for acceleration precomputed

The cubed-sphere model may be used to approximate any number of elements defined on a primary body. For example, it can approximate each component of acceleration, or the gravity potential. The accelerations are not directly derived from the potential, but are stored separately (in a submodel). Thus, for a model to provide both potential and three components of acceleration, values of all four parameters are stored at each point for future interpolation. In the following sections, any reference to modeling the gravity potential may also be applied to modeling acceleration (with the appropriate adjustments). Although the cubed-sphere model has been described in the literature [2], a more detailed description is included here for the purpose of clarity.

The cubed-sphere model is currently derived from an existing gravity model, hereafter called the base model. Although other models such as a polyhedron or mascon may be used, we currently use the spherical harmonic model as the base model. In the cubed-sphere model, the first four terms of the spherical harmonic expansion [i.e., the two-body term, J_2, and the $_{2;1}$ and $_{2;2}$ terms] are used directly. The cubed-sphere model does not include the lower-order terms to reduce the range of approximated values, reducing the cost of maintaining accuracy in the local model. The geopotential values computed by the remaining terms in the base model are then represented by the basis functions on the surface of the cube.

Temporal variations, such as solid or liquid tides, influence the geopotential. These variations mostly affect lower-degree terms of the potential. The cubed sphere only models terms of degree greater than or equal to a chosen minimum degree and order: in this case, 3. This parameter may be adjusted to allow for perturbations in the lower-degree terms, whereas higher-degree terms are expressed in the cubed-sphere formulation. Of course, this may slightly affect computation time.

A key parameter of the cubed-sphere model is the grid size N. Similar to the degree and order of the spherical harmonic model, the grid size defines the density of the grid on each cube face and is a measure of model fidelity. For a given altitude, the values of latitude and longitude are segmented such that

$$2 \quad x; \qquad 2 \quad y \tag{2}$$

where x and y are discrete values in the range [0, 1) with spacing N^{-1}.

relatively large variations in the gravity field as location changes. This is especially true along coastal and mountain regions. As discussed in [2], the cubed sphere was originally developed with multiresolution techniques in mind. However, adjusting the grid density to these levels reveals noise in the original spherical harmonics terms. The noise could be removed, which effectively modifies some higher-degree terms. Thus, the cubed sphere would no longer agree with the spherical harmonic model, which may currently cause resistance to its use. Additionally, early tests of the model for [2] demonstrated only small gains in speed as a result of such change and only a marginal decrease in memory required.

A user-specified number of nested concentric shells are required for interpolation in the radial direction. Shell spacing is determined by defining a set number of points (h_j) equally spaced in the interval [0, 1]. Shell locations are then

$$\frac{R}{r_j} \quad 1 \quad h_j^2 \tag{4}$$

where r_j

A given geopotential model must satisfy the Laplace equation:

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0 \tag{9}$$

Unfortunately, the direct calculation of the second derivatives within the cubed-sphere model from accelerations results in the loss of accuracy of one–two digits. To avoid such a loss of accuracy, values of the derivatives may be added to the model based on the variational equations of the spherical harmonic model. (Of course, this will almost double the file size.)

bottom subplot portrays the relative performance of trends in the integration-constant variations. For each orbit, we perform a linear fit mean at these lower altitudes, indicating a relatively small number of tests increase the mean value. Again, the models closely agree for higher altitudes, as indicated by the mean and median values with small error bars.

Results for the CS-162 model are provided in Fig. 7. Note that

Similarly, we compute the Fourier coefficients of g in terms of interpolating splines:

$$\hat{g}_n \quad \left(\frac{1}{N}\sum_{j \quad 0}^{N \quad 1} {}_j e^{\ 2 \ ijn/N}\right)\hat{L}_m\left(\frac{n}{N}\right) \quad \hat{}_n\hat{L}_m\left(\frac{n}{N}\right) \tag{A11}$$

The B-splines and the interpolating splines are related by (see, for example, [16])

$$\hat{L}_m\left(\frac{n}{N}\right) \quad \frac{\hat{B}_m \ \frac{n}{N}}{a \ \frac{n}{N}} \tag{A12}$$

where

$$a \ \omega \quad \sum_{j \in \mathbb{Z}} |\hat{B}_m \ \omega \quad j \ |^2 \tag{A13}$$