Department of Applied Mathematics
PROBABILITY AND STATISTICS PRELIMINARY EXAMINATION
August 2020

Instructions:

Do two of three problems in each section (Prob and Stat).
Place an X on the lines next to the problem numbers
that you are **NOT** submitting for grading.

Prob
1. ____
2. ____
3. ____

Do not write your name anywhere on this exam.
You will be identified only by your student number.
Write this number on each page submitted for grading.
Show all relevant work!

Stat
4. ____
5. ____
6. ____
Total ____

Student Number ____________________________

Probability Section

Problem 1.

(a) Let X be a non-negative continuous random variable with cdf F. Show that

$$E\left[\frac{1}{1+X}\right] = \int_0^\infty \frac{F(x)}{(1+x)^2}\,dx$$

(b) Let $X_1, X_2, \ldots, X_n$ be a random sample from the exponential distribution with rate 1. Define

$$Y_n = X_1 + \frac{1}{2}X_2 + \frac{1}{3}X_3 + \cdots + \frac{1}{n}X_n.$$

Find expressions for the moment generating functions for $X_{(n)}$ and Y_n. (Your expressions may contain sums or products but may not contain integrals.)

(c) Compare the two moment generating functions for $n = 1, 2$, and 3. Assuming that the pattern you see continues for all $n \geq 1$, what can you say about the distributions of $X_{(n)}$ and Y_n as they relate to each other?

Problem 2.

Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of iid random variables from the Poisson distribution with parameter .
Let $Y_n = X_n X_{2n}$ for $n \geq 1$ and consider the nth partial sum

$$S_n = Y_1 + Y_2 + \cdots + Y_n.$$

(a) Find $\mathrm{E}[S_n]$.

(b) Find a constant C, which may depend on but which may not depend on n, such that

$$Var[S_n] \leq Cn \quad \forall n \geq 1.$$

(c) Find a sequence of real numbers $\{a_n\}$ such that

$$\frac{S_n}{a_n} \xrightarrow{P} 1.$$

(Your a_n may depend on .)

Problem 3.

In a disease outbreak, there are three different states of an individual: the first state is "susceptible" (denote by s), the second is "infected" (denoted by i), and the third is "recovered" (denoted by r). The state of an individual at time $t \geq 0$, $X(t)$, is modeled as a continuous-time Markov chain with the infinitesimal generator (or rate matrix)

$$Q = \begin{bmatrix} & & 0 \\ 0 & & \\ 0 & & \end{bmatrix}, \tag{1}$$

for some $\ ;\ ;\ > 0$ with $\ <\ $. We assume that $X(0) = s$.

(a) Consider the first infection time $\tau_i := \inf\{t > 0 : X(t) = i\}$. Given $t > 0$, find the probability $P(\tau_i > t)$.

(b) Given $t > 0$, find the probability that $X(t) = i$ and the state r has not yet been visited, i.e.

$$P\left(X(t) = i \text{ and } X(u) \in \{s, i\}\ \forall u \in [0, t)\right).$$

(c) Given $t > 0$, what is the probability that the individual get infected three times during the period $[0, t]$?

(d) Suppose that there are $N > 0$ individuals in a population. Each individual is susceptible at time 0, and subject to the spread of the disease as in (1) *independently of other individuals*. As $t \to \infty$, what are the limiting fractions of population that are susceptible, infected, and recovered?

Statistics Section

Problem 4.

Consider $X_1; X_2; :::; X_n$ where X_i is exponentially distributed with mean $= _i$. Let $Y_1; Y_2; :::; Y_n$ be exponential random variables with $E[Y_i] = \quad _i$. Assume that the X's and Y's are all mutually independent.

In this problem, the parameters $; \ _1; \ _2; :::; \ _n$ are all positive and unknown.

(a) Find the maximum likelihood estimator (MLE) of .

For parts (b) and (c), assume that $_1; \ _2; :::; \ _n$ are known.

(b) Find the MLE for and the UMVUE (uniformly minimum variance unbiased estimator) for .

(c) Compute the relative e ciency of your estimators from part (b). What can you say as $n \ ! \ 1$?

Problem 5.

Suppose that X and Y are iid $N(0; 1)$ random variables. It is well known that X^2 and Y^2 each have a $^2(1)$ distribution.

(a) Let $W = \min(X; Y)$. Show that $W^2 \quad ^2(1)$.

(b) Now suppose that X and Y are iid $N(\ ; \ ^2)$ random variables with known and ^{2} unknown. Use part (a) to derive a 100(1)% con dence interval for ^{2} based on the statistic $W = \min(X; Y)$.

Problem 6.

Suppose that we have a random sample, $X_1; X_2; :::; X_n$ from the distribution with pdf

$$f(x; \ ) = \frac{1}{6 \ ^3} x^2 e^{\ x=} \ I_{(0; 1)}(x)$$

(a) Find the best (most powerful) test of size of $H_0 : \ = \ _0$ versus $H_1 : \ = \ _1$, assuming that $_1 > \ _0$. Give your answer in terms of a chi-squared critical value.

(b) Is your test uniformly most powerful (UMP) for the alternative hypothesis $H_1 : \ > \ _0$? Explain.

(c) Is your test uniformly most powerful (UMP) for the alternative hypothesis $H_1 : \ \neq \ _0$? Explain.

(d) Derive an approximate large-sample generalized likelihood ratio test (GLRT) of size for the hypotheses in parts (b) and (c) if your test was not a UMP test. (Note: Depending on how you answered (b) and (c), you may have nothing to do here, you may have one test to do, or you may have 2 tests to do.)