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A[illegible] We review the methods in [4] and [24] for constructing quadratures for bandlimited exponentials and introduce a new algorithm for the same purpose. As in [4], our approach also yields generalized Gaussian quadratures for exponentials integrated against a non-sign-definite weight function. In addition, we compute quadrature weights via ℓ^2 and ℓ^∞ minimization and compare the corresponding quadrature errors.

1. [illegible]

We revisit the construction of quadratures for bandlimited exponentials $\left\{e^{ibx}\right\}_{|b|\leq c}$ integrated against a real-valued weight function w on the interval $|x| \leq 1$. These functions are not necessarily periodic in $[-1,1]$. Unlike the classical Gaussian quadratures for polynomials which integrate exactly a subspace of polynomials up to a fixed degree, Gaussian type quadratures for exponentials use a finite set of nodes in order to integrate the infinite set of functions $\left\{e^{ibx}\right\}_{|b|\leq c}$. While it is not possible to construct exact quadratures in this case, those introduced in [4] integrate with (user-selected) accuracy ϵ all exponentials with $|b| \leq c$. We note that, for a given bandlimit c and accuracy ϵ, quadratures of this type are not unique.

For a given accuracy ϵ, bandlimit c, and weight function w, the Gaussian-type quadratures in [4] are designed to integrate functions in the linear s

of weights further. A Newton-type optimization (using ℓ^2 norm) is shown to gain an extra $1-2$ digits in the accuracy of the quadratures.

A drawback of this approach is that it is not clear how to apply it for a general weight function since no differential operator is available (see [14]). On the other hand, given that a differential operator is available for the weight function $w = 1$, positions of nodes may be found rapidly in $O(M)$ operations using the algorithm in [10]. This fact that the PSWFs satisfy the second order differential equation in (2.2) implies that their zeros may be found without ever explicitly computing the functions themselves. We note that the DPSWFs (see previous section) also satisfy a second order differential equation and, hence, the algorithm in [10] is applicable in that case as well.

3. [illegible]

3.1. [illegible] Let us illustrate finding nodes as eigenvalues of a matrix by constructing the classical Gaussian quadrature with M nodes $\{x_m\}_{m=1}^{M}$. Let us consider a basis $\{\varphi_l(x)\}_{l=0}^{M-1}$ in the subspace of real-valued polynomials of degree up to $M-1$ equipped with the inner product

$$\langle p,q\rangle = \int_{-1}^{1} p(x)q(x)\,w(x)dx.$$

We form the square matrix $A \in \mathbb{R}^{M\times M}$ of entries

$$A_{ll'} = \int_{-1}^{1} \varphi_l(x)\varphi_{l'}(x)\,w(x)dx = \sum_{m=1}^{M} \varphi_l(x_m)w_m\varphi_{l'}(x_m),$$

where x_m are the desired quadrature nodes and w_m the corresponding quadrature weights. Since the product of two polynomials in this subspace has degree of at most $2M -$

quadratures for an arbitrary user-selected accuracy ϵ. These quadratures integrate exponentials of bandlimit c against a real-valued weight function $w(x)$, so that

$$\left|\int_{-1}^{1} e^{ibx} w(x)\,dx - \sum_{m=1}^{M} e^{ibx_m} w_m\right| < \epsilon, \quad |b| \leq c, \tag{3.1}$$

where $x_m \in [-1,1]$ and $w_m \in \mathbb{R} \setminus \{0\}$.

To solve this problem, we consider

$$G(b, b\ ) = \int_{-1}^{1} e^{i\frac{b}{2}x} e^{-i\frac{b'}{2}x} w(x)dx, \quad |b|, |b\ | \leq c, \tag{3.2}$$

which we discretize as

$$\int_{-1}^{1} e^{i\frac{c}{2}\frac{n}{N}x} e^{-i\frac{c}{2}\frac{n'}{N}x} w(x)dx, \quad n, n\ = -N, \ldots, N, \tag{3.3}$$

where $N > M$ by an (oversampling) factor. However, it is more convenient to consider instead the Hermitian $(N+1) \times (N+1)$ matrix

$$_{nn'} = \int_{-1}^{1} e^{ic\frac{n}{N}x} e^{-ic\frac{n'}{N}x} w(x)dx, \quad n, n\ = 0, \ldots, N, \tag{3.4}$$

which oversamples the interval $[-c, c]$ in the same fashion with an appropriate N. Note that if $w \geq 0$, is a Gram matrix of inner products. As discussed in Section 2.1, the resulting quadratures also depend weakly on the choice of N.

Let us seek $\{x_m\}_{m=1}^{M}$ and $\{w_m\}_{m=1}^{M}$, with $M < N$, so that

$$|\ _{nn'} - \ _{nn'}| < \epsilon, \quad n, n\ = 0, \ldots, N, \tag{3.5}$$

where the quadrature matrix has entries

$$_{nn'} = \sum_{m=1}^{M} e^{icx_m\frac{n}{N}} w_m e^{-icx_m\frac{n'}{N}}, \quad n, n\ = 0, \ldots, N. \tag{3.6}$$

First, we show that it is possible to obtain the quadrature nodes by finding eigenvalues of an appropriate matrix. We consider two submatrices of , A n

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(2) Take the SVD of , = , and select the index M corresponding to the singular value σ_M such that σ_M/σ_0 is close to the desired accuracy ϵ.

(3) Truncate the matrix (such that it contains the singular vectors corresponding to the singular values $\sigma_0, \ldots, \sigma_{M-1}$) and form the matrices $_M$ and $_M$ from equation (3.12).

(4) Using the pseudo-inverse, form the matrix $\mathrm{C}_M = {}_M {}_M$ and find its eigenvalues, $\left\{e^{icx_m/N}\right\}_{m=1}^{M}$, from which we extract the nodes x_m, $m = 1, \ldots, M$.

Remark 1. Similar to the algorithms for finding quadratures in [4] and [24], if we compute high accuracy quadratures (e.g., $\epsilon < 10^{-12}$), we need to use extended precision arithmetic in our computations. Once the quadrature nodes and weights are obtained, no extra precision is needed for their use.

Remark 2. Algorithm 1 requires $O\left(M^3\right)$ operations and is applicable to general weight functions (see examples below).

Remark 3. The explicit introduction of inner products (if applied to the case of decaying exponentials) provides an interpretation of the so-called HSVD [19] or the matrix-pencil method [15, 16, 17] algorithms (that are essentially the same). In our view, our approach simplifies the understanding of these algorithms originally introduced in electrical engineering literature as a sequence of steps similar to those in Algorithm 1.

4. [illegible]

We calculate quadrature weights using two different approaches: standard least squares and residual minimization. The most straightforward approach is to use least squares. However, we may achieve a better maximum error if we use residual minimization. This approach leads us to set up the problem as a second order cone program (since our matrices are complex), and then apply an appropriate solver (see Section 8.2).

4.1. [illegible]

To find the weights w_m, $m = 1, \ldots, M$ that satisfy (3.1), we solve a rectangular Vandermonde system using least squares. The Vandermonde matrix $\in \mathbb{C}^{(2N+1)\ M}$ is defined as $_{nm} = e^{icx_m n/N}$, where x_m, $m = 1 \ldots M$, are the quadrature nodes, c is the bandlimit parameter and $n = -N, \ldots, N$. We solve the overdetermined system $\vec{w} = \vec{u}$, where $\vec{w} = \{w_m\}_{m=1}^{M}$ is the vector of weights and $\vec{u} = \{u_n\}_{n=-N}^{N}$ is the vector of trigonometric moments

$$u_n = u\left(\frac{n}{N}\right) = \int_{-1}^{1} e^{icx\frac{n}{N}} w(x)\, dx.$$

The performance of our quadrature nodes using least squares weights is illustrated in Table 2 and Figure 6.3(a).

This approach to finding weights is related to the method used in [4] since we also solve a Vandermonde system. However, in [4] the Vandermonde system size may vary between $M \times M$ and $(N+1) \times (N+1)$. The different sizes of the Vandermonde system are due to the knowledge, or lack thereof, of the general location of the nodes. If the nodes are known to belong to a particular subset of the unit circle,

5. [illegible]

5.1. An [illegible] Let us find quadrature nodes for the integral

$$u^{(c)}(B,l,\cos\theta) = \frac{1}{2}\int_{-1}^{1} I_0\left(\pi B\sqrt{1-\left(\frac{x}{l}\right)^2}\right) e^{icx\cos(\theta)}dx, \tag{5.1}$$

where c is the bandlimit and I_0 is the modified Bessel function of order zero. This integral arises in antenna design and, for parameters $l = 1$ and $B = 1$, a quadrature for (5.1) is computed in [7, Eq. 6.7] by a different approach. However, our approach is simpler and yields similar results. Given the weight function

$$w(x) = I_0\left(\pi\sqrt{1-x^2}\right), \tag{5.2}$$

we obtain its trigonometric moments as

$$u_n^{(c)} = \frac{1}{2}\int_{-1}^{1} e^{icxn/N} w(x)\,dx = \operatorname{sinc}\left(\sqrt{(c\frac{n}{N})^2-\pi^2}\right), \quad n = -N,\dots N, \tag{5.3}$$

corresponding (up to a factor) to the samples of the radiation pattern. Identity (5.3) may be obtained extending formula 6.616.5 in [12, p. 698]. We also note that the weight function (5.2) is a scaled version of the so-called Kaiser window (see e.g. [18]).

We form

$$_{nn'} = u^{(c)}_{n-n'}, \quad n, n = 0,\dots N,$$

with $N = 252$ and $c = 10\pi$, and use Algorithm 1 in Section 3.3. We truncate the SVD of the matrix at the (normalized) singular value σ_{22}, $\sigma_{22}/\sigma_0 \approx 1.2\cdot 10^{-15}$, yielding 22 quadrature nodes. Using the residual minimization (see Section 4.2), we compute the weights resulting in a quadrature with maximum absolute error $\epsilon = 1.21\cdot 10^{-14}$. We verify the accuracy of this quadrature numerically and illustrate the result in Figure 5.1. This quadrature should be compared with that corresponding to the bandlimit 20π in [7, Table 6.3] since we integrate on $[-1,1]$ instead of $[-1/2,1/2]$ as in [7].

5.2. [illegible] We demonstrate that our method yields quadratures for weight functions w that are not sign-definite. For the weight function

$$w(x) = (x-1/10)\cdot e^{-(3\pi x/5-1/5)^2} + 1/(5e), \tag{5.4}$$

we calculate the nodes and weights for the bandlimit $c = 5\pi$, choosing $N = 127$ and the singular value $\sigma_{14}/\sigma_0 = 5.0\cdot 10^{-14}$. Figure 5.2(a) illustrates the weight function $w(x)$, $x \in [-1,1]$, and Figure 5.2(b) shows that the weights of the quadrature follow the shape of the weight function $w(x)$. The error of the quadrature with 14 nodes and weights is illustrated in Figure 5.2(c), where the maximum error is $6.68\cdot 10^{-14}$.

We note that the approach in [4] also allows us to obtain quadratures for weight functions $w(x)$ that are not sign-definite as is shown in [2].

Quadrature nodes and weights for c = 50		
Nodes	ℓ^2 min weights	ℓ^∞ min weights
0.05098496373726	$1.0194136874164 \cdot 10^{-1}$	$1.0194136790749 \cdot 10^{-1}$

Note that our approach is more general since it may be applied to any basis $\{p_n(x)\}_{n=1}^{N}$, even if it is not orthogonal (no 3-term recurrence is available); it also generalizes to other sets of functions or non-positive weights.

8.2. [illegible] We review the primal-dual interior-point method of [20], the algorithm we implemented in extended precision to compare with the results obtain

8.2.2. Primal-dual interior-point method. The primal-dual interior-point algorithm solves (8.3) by minimizing the difference between the primary and the dual objective functions, known as the duality gap,

$$\eta(\;,\;,\;) = {}^{t}\; + \sum_{i=1}^{N} \left({}^{t}_{i}\; {}_{i} + d_i w_i \right).$$

This gap is non-negative for feasible $\;,\;,\;$. Considering strictly feasible primal and dual problems (i.e., the inequalities in (8.3) and (8.4) are replaced by strict inequalities), we know that there exists solutions where the duality gap $\eta(\;,\;,\;) =$

Finally, we state the algorithm. Given strictly feasible initial points (, ,), a tolerance □ > 0, and the parameter ν ≥ 1, we

(1) Solve equation (8.5) for the primal and dual search directions.
(2) Perform a plane search to find the (p, q) that minimize ϕ (+ pδ , + qδ , + qδ).
(3) Update = +pδ , = +qδ , and = +qδ as long as η (, ,) > □.

We note that as η decreases in size the system of equations (8.5) becomes ill conditioned, which results in indeterminate search directions.

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